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Approximation on Universal Algebras by Means of Polynomial Functions

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1. INTRODUCTION

The interpolation of real functions on finitely many points by means of polynomial functions plays an important role in the theory of approximation of functions. This interpolation property can be investigated also for arbitrary universal algebras, as has been done by Kaiser [9]. A topological view on interpolation leads to the concept of local approximation property: a topological universal algebra A has this property if the values of any function $f: A \to A$ on finitely many points can be approximated "arbitrarily closely" by values of polynomial functions on these points. The study of such algebras has been initiated in Kowol [10].

In this paper we study the local approximation property for k-place functions $f: A^k \to A$ by a method due to Maurer and Rhodes [13] and generalized by Istinger and Kaiser [6]. Among others, it allows us to describe all topological groups, rings, and near-rings having this property, thereby obtaining some results of Istinger and Kaiser [6] and Kaiser [8], respectively, as special cases (the finite and discrete case, respectively). Our description shows a close connection between the local approximation property and density properties studied by Jacobson (for rings, [7]) and Betsch (for near-rings, [1]). As an application, we derive a classical density theorem for rings of linear transformations of vectorspaces over skewfields due to Jacobson [7].

2. PRELIMINARIES

Let $A = \langle A, \Omega \rangle$ be a universal algebra and k a positive integer. On the set $F_k(A)$ of all k-place functions on A with values in A, we define the operations of A pointwise, thus obtaining the full k-place function algebra

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$F_k(A) = \langle F_k(A), \Omega \rangle$. The subalgebra $P_k(A)$ of $F_k(A)$ generated by the constant functions and the k projections is called the algebra of k-place polynomial functions on A (see Lausch and Nöbauer [12, Chap. 1]). We endow A with a topology $\mathfrak{T}$ which is compatible with any n-ary operation of Ω $(n > 0)$, and obtain in that way a topological universal algebra $A = \langle A, \Omega, \mathfrak{T} \rangle$. Endowing $F_k(A) = A^{(A^k)}$ with the product topology $\mathfrak{T}^*$, we get the topologized full k-place function algebra $F_k(A) = \langle F_k(A), \Omega, \mathfrak{T}^* \rangle$.

DEFINITION. A topological universal algebra A has the *k-local approximation property* if $P_k(A)$ is dense in $F_k(A)$.

If an arbitrary universal algebra A is endowed with the indiscrete topology, it always has the k-local approximation property for all k, since $F_k(A)$ is endowed with the indiscrete topology too. To avoid such trivial cases, we always suppose the following:

The topology on A satisfies the Hausdorff separating axiom T_2.

Remarks. (1) It is easy to verify that the set $C_k(A)$ of continuous functions of A^k to A is a subalgebra of $F_k(A) = \langle F_k(A), \Omega \rangle$ the operations being defined pointwise. Since the constant functions and the projections are continuous, we have $P_k(A) \subseteq C_k(A)$.

(2) If A is endowed with the discrete topology we obtain the concept of k-interpolation property (k-locally completeness), see [9].

(3) If A is finite, we have: A has the k-local approximation property iff it has the k-interpolation property iff $P_k(A) = F_k(A)$ (that is, A is polynomially complete, see [12, Chap. 1]), because A finite and T_2 implies $F_k(A)$ finite and T_2 and thus discrete.

PROPOSITION 1. *Let A be a topological T_2 universal algebra satisfying the k-local approximation property. Then A also has l-local approximation property for $l < k$.*

Proof. Let $f \in F_l(A)$ and let U_l be a neighborhood of f. We have to show the existence of $q \in P_l(A)$ with $q \in U_l$. We may suppose that U_l is element of the basis of the neighborhoods of f, that is $U_l = \prod_{i \in A^l} O_i$ with $O_i = A$ for all $i \in A^l$ except $i = i_1, \dots, i_r$, O_i open in A and $f(i) \in O_i$, $i \in A^l$. We fix elements $b_{l+1}, \dots, b_k \in A$ and set $j_\alpha = (i_\alpha, b_{l+1}, \dots, b_k) \in A^k$, $\alpha = 1, \dots, r$. We define $\varphi \in F_k(A)$ by $\varphi(a_1, \dots, a_k) = f(a_1, \dots, a_l)$ and V_k by $V_k = \prod_{j \in A^k} Q_j$ where $Q_{j_\alpha} = O_{i_\alpha}$ for $\alpha = 1, \dots, r$ and $Q_j = A$ otherwise. Evidently, V_k is an open neighborhood of φ. By assumption there exists a $p \in P_k(A)$ with $p \in V_k$. If a representing word of p has the form $w(a_m, \xi_1, \dots, \xi_k)$ (see [12, Chap. 1, Proposition 6.41]), then $q = w(a_m, \xi_1, \dots, \xi_l, b_{l+1}, \dots, b_k) \in P_l(A)$ and fulfills $q \in U_l$.

PROPOSITION 2. *Let A be a topological T_2 universal algebra having the*

2-local approximation property. Then A also has the k-local approximation property for all positive integers k.

Proof. If A is finite, then by Remark 3 the k-local approximation property coincides with k-polynomially completeness, hence we can apply Theorem 1.11.2 of [12]. Therefore, we may assume A to be infinite and prove the proposition by induction on k. The 1- and 2-local approximation property follows from Proposition 1 and assumption, respectively. We assume $k \geqq 3$. Let $f \in F_k(A)$ and U_k be an open neighborhood of f of the form $U_k = \prod_{i \in A^k} O_i$, where O_{i_α} is open in A, $\alpha = 1,..., r$, $O_i = A$ otherwise and $f(i) \in O_i$ for all $i \in A^k$. Since A is infinite, there exists a bijective function $\psi: A^{k-1} \to A$. Defining $\chi: A^2 \to A$ by $\chi(u, v) = f(\psi^{-1}(u), v)$ we obtain $\chi(\psi(a_1,..., a_{k-1}), a_k) = f(a_1,..., a_k)$ for all $(a_1,..., a_k) \in A^k$. We set $j_\alpha = (i_{\alpha,1},..., i_{\alpha,k-1})$, $\alpha = 1,..., r$, and define V_2 by $V_2 = \prod_{i \in A^2} P_i$, where $P_i = A$ for all $i \in A$, $i \neq (\psi(j_\alpha), i_{\alpha,k})$, $\alpha = 1,..., r$, and $P_{(\psi(j_\alpha), i_{\alpha,k})} = O_{i_\alpha}$. Then V_2 is an open neighborhood of χ because $\chi(\psi(j_\alpha), i_{\alpha,k}) = \chi(\psi(i_{\alpha,1},..., i_{\alpha,k-1}), i_{\alpha,k}) = f(i_\alpha) \in O_{i_\alpha}$, $\alpha = 1,..., r$. By assumption there exists a $p \in P_2(A)$ satisfying $p \in V_2$ which implies in particular $p(\psi(i_{\alpha,1},..., i_{\alpha,k-1}), i_{\alpha,k}) \in O_{i_\alpha}$ for all α. According to Remark 1, p is continuous. Therefore, the inverse image $p^*(O_{i_\alpha})$ of O_{i_α} is an open set in A^2 containing the point $c_\alpha = (\psi(j_\alpha), i_{\alpha,k})$, $\alpha = 1,..., r$. Thus there exist open sets Q_{1,i_α}, Q_{2,i_α} in A satisfying $Q_{i_\alpha} = Q_{1,i_\alpha} \times O_{2,i_\alpha} \subseteqq p^*(O_{i_\alpha})$, hence $p(Q_{i_\alpha}) \subseteqq O_{i_\alpha}$, and $\psi(j_\alpha) \in Q_{1,i_\alpha}$, $i_{\alpha,k} \in Q_{2,i_\alpha}$ $(\alpha = 1,..., r)$, respectively. Finally we define W_{k-1} by $W_{k-1} = \prod_{i \in A^{k-1}} R_i$, where $R_i = A$ for all $i \in A^{k-1}$, $i \neq j_1,..., j_r$, and $R_{j_\alpha} = Q_{1,i_\alpha}$, $\alpha = 1,..., r$. By construction W_{k-1} is an open neighborhood of ψ. Now $\psi \in F_{k-1}(A)$ implies the existence of a polynomial function $q \in P_{k-1}(A)$ satisfying $q \in W_{k-1}$ (induction hypothesis). If $p = w_1(a_m, \xi_1, \xi_k)$ and $q = w_2(b_1, \xi_1,..., \xi_{k-1})$ are representations of p and q by words (see [12, Chap. 1, Prop. 6.41]), we get $p \circ q = w_1(a_m, w_2(b_1, \xi_1,..., \xi_{k-1}), \xi_k)$ is an element of $P_k(A)$. $q \in W_{k-1}$ implies, especially, $q(i_{\alpha,1},..., i_{\alpha,k-1}) \in Q_{1,i_\alpha}$, $\alpha = 1,..., r$, and therefore we derive

$$\begin{aligned}(p \circ q)(i_\alpha) &= (p \circ q)(i_{\alpha,1},..., i_{\alpha,k}) \\ &= w_1(a_m, q(i_{\alpha,1},..., i_{\alpha,k-1}), i_{\alpha,k}) \in p(Q_{i_\alpha}) \subseteqq O_{i_\alpha}\end{aligned}$$

for all α, which means $p \circ q \in U_k$.

Propositions 1 and 2 yield three possible cases for an arbitrary topological universal algebra A:

(i) A has the k-local approximation property for all k, which will be abbreviated by saying A has the local approximation property.

(ii) A has the k-local approximation property only for $k = 1$.

(iii) A has the k-local approximation property for no k.

Concerning case (ii) we showed in [10, Satz 1] that this property is hereditary to dense subalgebras of A. An analogous result holds in general the proof of which we will omit since it can be easily generalized.

THEOREM 1. *Let A be a topological T_2 universal algebra having the k-local approximation property for any k. Then every dense subalgebra of A endowed with the induced topology also has the k-local approximation property.*

3. DESCRIPTION OF ALGEBRAS HAVING THE k-LOCAL APPROXIMATION PROPERTY

We start with the following:

LEMMA 1. *A topological T_2 lattice V with more than one element never has the k-local approximation property for any k.*

Proof. Follows from Proposition 1 and Lemma 3 of [10].

In the following we will use Satz 2 of [10] and state it in a modified form for the sake of completeness.

THEOREM 2. *Let V be a variety and $A = \langle A, \Omega, \mathfrak{T} \rangle$ a topological T_2 universal algebra with the following properties: There exist $0 \in A$, $\eta, \mu \in P_2(A)$ such that*

(1) $\mu(0, a) = \mu(a, 0) = 0$ *for all* $a \in A$,

(2) $\eta(\mu(a, b), b) = a$ *for all* $a, b \in A$.

Let κ be an arbitrary congruence relation on A, I the congruence class containing 0, and let κ satisfy the property

$$\omega(\mu(i_1, a_1),..., \mu(i_n, a_n)) \in \{\mu(i, \omega(a_1,..., a_n)), i \in I\} \tag{1}$$

for all $a_1,..., a_n \in A$, $i_1,..., i_n \in I$ and all n-ary operations ($n \geqq 1$), $\omega \in \Omega$. If A has the k-local approximation property for any k and $I \neq \{0\}$ then I is dense in A.

Proof. Using Bemerkung (2) to Hilfssatz in [10] we derive from the assumptions for an arbitrary polynomial function $p \in P(A)$,

$$p(i) = \mu(j(i), p(0)) \qquad \text{for all } i \in I \text{ with some } j(i) \in I. \tag{2}$$

Since condition (2) guarantees that we can express $j(i)$ in terms of $p(i)$, $p(0)$ ($j(i) = \eta(\mu(j(i), p(0)), p(0)) = \eta(p(i), p(0))$) we can use the proof of Satz 2 in [10] word by word to show the theorem.

This theorem can be applied to a lot of well-known varieties, e.g., to the variety of groups, rings, near-rings, and more generally to the variety of multioperatorgroups (see [11] for definition), since (1) always is fulfilled:

COROLLARY 1. *Let V be any of the above mentioned varieties and A a topological T_2 universal algebra of V having the k-local approximation property for any k. Then all non-trivial ideals (normal subgroups, respectively) are dense in A.*

The reverse statement will be treated in the next theorem; the proof of which is based on ideas used in Istinger and Kaiser [6], originally due to Maurer and Rhodes [13].

THEOREM 3. *Let A be a topological T_2 universal algebra having the following properties: There exist $0 \in A$, $\varphi, \psi \in P_2(A)$ and $\chi \in P_n(A)$, $n \geqq 2$, such that*

(1) $\varphi(a, a) = 0$ *for all* $a \in A$.

(2α) $\psi(0, a) = \psi(a, 0) = a$ *for all* $a \in A$.

(β) $\psi(\varphi(a, b), b) = a$ *for all* $a, b \in A$.

(3α) χ *is not constant.*

(β) $\chi(a_1,..., a_n) = 0$ *if at least one of the elements* a_i *equals* 0.

If for all non-trivial congruences the congruence class containing 0 is dense in A, then A has the local approximation property.

Proof. By Proposition 2 it suffices to show that A has the 2-local approximation property. Let f be an arbitrary element of $F_2(A)$ and let $O = \prod_{i \in A^2} O_i$ be an element of the basis of the open neighborhoods of f, that is, O_i is open in A for all $i \in A^2$, $O_i = A$ for all i except finitely many which we will denote by $i_1,..., i_r$, and $f(i) \in O_i$, $i \in A^2$. We have to show that there exists a $p \in P_2(A)$ satisfying $p \in O$. First we remark that it suffices to show the existence of $p_{i_\alpha} \in P_2(A)$ with $p_{i_\alpha}(i_\alpha) \in O_{i_\alpha}$, $\alpha = 1,..., r$ and $p_{i_\alpha}(i_\beta) = 0$ for $\beta \neq \alpha$. Because then the polynomial function

$$p = \psi(\cdots (\psi(\psi(p_{i_1}, p_{i_2}), p_{i_3}) \cdots p_{i_r})$$

fulfills $p(i_\alpha) = p_{i_\alpha}(i_\alpha) \in O_{o_\alpha}$, $\alpha = 1,..., r$ using 2α) several times. Because of symmetry, we can furthermore restrict ourselves to the case $\alpha = 1$ so that our problem is reduced to find for any O_{i_1} open in A a polynomial function $p_{i_1} \in P_2(A)$ with $p_{i_1}(i_1) \in O_{i_1}$ and $p_{i_1}(i_\beta) = 0$ for $\beta = 2,..., r$.

To do this, we define sets I_γ in the following way:

$$I_\gamma = \{a \in A, \exists u \in P_2(A) \text{ with } u(i_1) = a, u(i_1) = \cdots = u(i_\gamma) = 0\}, \qquad 1 \leqq \gamma \leqq r.$$

$I_\gamma \neq \varnothing$ since $0 \in I_\gamma$ for all γ (take u the constant polynomial function with value 0). We will show that I_γ is dense in A for all $\gamma = 1,\ldots, r$. This will finish the proof of our theorem because then there exists an element $a \in O_{i_1} \cap I_r$ and by definition of I_r a $u \in P_2(A)$ with $u(i_1) = a \in O_{i_1}$, $u(i_2) = \cdots = u(i_r) = 0$, which is what we wanted to show.

Therefore all we need to prove is the following:

LEMMA 2. *I_γ is dense in A for all $\gamma = 1,\ldots, r$.*

The proof of this result depends on

LEMMA 3. *Let γ be a fixed element of $\{1,\ldots, r\}$. The relation θ_γ defined by $a\theta_\gamma b$ iff $\varphi(a, b) \in I_\gamma$ is a congruence relation on A.*

Proof of Lemma 3. (1) $a\theta_\gamma a$ holds because $\varphi(a, a) = 0$ for all $a \in A$ by (1).

(2) Let $a\theta_\gamma b$ that is $\varphi(a, b) \in I_\gamma$. Hence, there exists a $u \in P_2(A)$ with $u(i_1) = \varphi(a, b)$ and $u(i_\alpha) = 0$ for $\alpha = 2,\ldots, \gamma$. Denoting by λ_α the constant polynomial function of $P_2(A)$ with value a we define the polynomial function q by $q = \varphi(\lambda_b, \psi(u, \lambda_b))$. Then $q \in P_2(A)$ and we have, using the assumptions (2β), (2α), and (1),

$$q(i_1) = \varphi(b, \psi(\varphi(a, b), b)) = \varphi(b, a)$$
$$q(i_\alpha) = \varphi(b, b) = 0 \text{ for } \alpha = 2,\ldots, \gamma.$$

Thus, by definition, $\varphi(b, a) \in I_\gamma$, which implies $b\theta_\gamma a$.

(3) To prove transitivity, we assume $a\theta_\gamma b$, $b\theta_\gamma c$. By the symmetry of θ_γ we may also assume $c\theta_\gamma b$. Thus there exist polynomial functions $u, v \in P_2(A)$ with $u(i_1) = \varphi(a, b)$, $v(i_1) = \varphi(c, b)$, and $u(i_\alpha) = v(i_\alpha) = 0$ for $\alpha = 2,\ldots, \gamma$. We set $w = \varphi(\psi(u, \lambda_b), \psi(v, \lambda_b))$. Then $w \in P_2(A)$ and we have, using (2β), (2α), and (1) again,

$$w(i_1) = \varphi(\psi(\varphi(a, b), b), \psi(\varphi(c, b), b) = \varphi(a, c)$$
$$w(i_\alpha) = \varphi(\psi(0, b), \psi(0, b)) = \varphi(b, b) = 0 \qquad \text{for all} \quad \alpha = 2,\ldots, \gamma.$$

Hence $\varphi(a, c) \in I_\gamma$ and $a\theta_\gamma c$ holds.

(4) Let ω be an $m$ary operation and $a_l\theta_\gamma b_l$ for $l = 1,\ldots, m$. There are $u_l \in P_2(A)$ with $u_l(i_1) = \varphi(a_l, b_l)$, and $u_l(i_\alpha) = 0$ for all l and $\alpha = 2,\ldots, \gamma$. We set

$$q = \varphi(\omega(\psi(u_1, \lambda_{b_1}),\ldots, \psi(u_m, \lambda_{b_m}), \lambda_{\omega b_1 \cdots b_m}).$$

Then $q \in {}_2(A)$ and

$$q(i_1) = \varphi(\omega(\psi(\varphi(a_1, b_1), b_1), \dots, \psi(\varphi(a_m, b_m), b_m), \omega b_1 \cdots b_m)$$
$$= \varphi(\omega a_1 \cdots a_m, \omega b_1 \cdots b_m)$$
$$q(i_\alpha) = \varphi(\omega(\psi(0, b_1), \dots, \psi(0, b_m), \omega b_1 \cdots b_m)$$
$$= \varphi(\omega b_1 \cdots b_m, \omega b_1 \cdots b_m) = 0 \qquad \text{for} \quad \alpha = 2, \dots, \gamma.$$

Hence $\omega a_1 \cdots a_m \theta_\gamma \omega b_1 \cdots b_m$, which proves Lemma 3.

Proof of Lemma 2. We use induction on γ. If $\gamma = 1$, then $I_\gamma = A$ since all constant functions are elements of $P_2(A)$.

Now consider the case $\gamma = 2$. Since $i_\delta = (i_{\delta 1}, i_{\delta 2}) \in A^2$, $\delta = 1, 2$, and $i_1 \neq i_2$, there exists an index—without loss of generality we assume the index to be 1—with $i_{11} \neq i_{12}$. We denote by π_1 the projection $\pi_1(a_1, a_2) = a_1$ $(a_1, a_2 \in A)$ and put $q = \varphi(\varphi(\pi_1, \lambda_{i_{12}}), \lambda_0)$. Then $q \in P_2(A)$ and

$$q(i_1) = \varphi(\varphi(i_{11}, i_{12}), 0)$$
$$q(i_2) = \varphi(\varphi(i_{12}, i_{12}), 0) = \varphi(0, 0) = 0.$$

This implies $\varphi(i_{11}, i_{12})\, \theta_\gamma 0$, where $\varphi(i_{11}, i_{12}) \neq 0$, because otherwise we would derive by $(2\alpha, \beta)$ $i_{11} = \psi(\varphi(i_{11}, i_{12}), i_{12}) = \psi(0, i_{12}) = i_{12}$, a contradiction. The assumption of the theorem implies that the congruence class $B = \{b \in A, 0\theta_\gamma b\} = \{b \in A,\ \varphi(b, 0) \in I_\gamma\}$. Now $\varphi(a, 0) = a$ for all $a \in A$ by setting $b = 0$ in (2β) and using (2α), so $B = I_\gamma$ and therefore Lemma 2 is valid in the case $\gamma = 2$.

In the case $\gamma \geqq 3$ we consider the sets

$$I_{1,\gamma-1} = \{a_1 \in A, \exists u_1 \in P_2(A) \text{ with } u_1(i_1) = a_1, u_1(i_3)$$
$$= \cdots = u_1(i_\gamma) = 0\}$$
$$I_{\alpha,\gamma-1} = \{a_\alpha \in A, \exists u_\alpha \in P_2(A) \text{ with } u_\alpha(i_1) = a,\ u_\alpha(i_2)$$
$$= u_\alpha(i_4) = \cdots = u_\alpha(i_\gamma) = 0\}, \qquad \alpha = 2, \dots, n.$$

According to induction hypothesis $I_{\beta,\gamma-1}$ are dense in A for all $\beta = 1, \dots, n$. Suppose $\chi(a_1, \dots, a_n) = 0$ for all $a_\beta \in I_{\beta,\gamma-1}$. Since χ is continuous (Remark 1), A is hausdorff, and $I_{1,\gamma-1} \times \cdots \times I_{n,\gamma-1}$ is dense in A^n, we get by the principle of extension of identities (see [2, Section 1.8.1, Corollary 1]) that χ is the constant function with value 0, contrary to assumption (3α). Therefore, there exist $b_\beta \in I_{\beta,\gamma-1}$ and belonging $v_\beta \in P_2(A)$, $\beta = 1, \dots, n$, with $\chi(b_1, \dots, b_n) \neq 0$. We set $q = \varphi(\chi(v_1, \dots, v_n), \lambda_0)$, then $q \in P_2(A)$ and

$$q(i_1) = \varphi(\chi(v_1(i_1), \dots, v_n(i_1), 0) = \varphi(\chi(b_1, \dots, b_n), 0)$$

and because of (3β),

$$q(i_\alpha) = \varphi(\chi(v_1(i_\alpha),\dots, v_n(i_\alpha), 0) = \varphi(0, 0) = 0 \quad \text{for all } \alpha = 2,\dots, n$$

since at least one of the values $v_\beta(i_\alpha) = 0$, $\beta = 1,\dots, n$. Therefore we have found an element $\chi(b_1,\dots, b_n) \neq 0$ with $\chi(b_1,\dots, b_n)\, \theta_\gamma 0$ and, continuing as in the case $\gamma = 2$, this proves Lemma 2.

Combining Theorem 2 and 3 we obtain

THEOREM 4. *Let V be a variety and $A = \langle A, \Omega, \mathfrak{T}\rangle$ a topological T_2 universal algebra having the following properties: There exist $0 \in A$, $\varphi, \psi \in P_2(A)$, $\chi \in P_n(A)$, $n \geqslant 2$, such that*

(1) $\psi(0, a) = \psi(a, 0) = a$ *for all* $a \in A$,

(2) $\varphi(\psi(a, b), b) = a$, $\psi(\varphi(a, b)) = b$ *for all* $a, b \in A$ (*or symmetrically*: (2′) $\varphi(a, \psi(a, b)) = b$, $\psi(a, \varphi(a, b)) = b$ *for all* $a, b \in A$,

(3) $\chi \not\equiv 0$ *and* $\chi(a_1,\dots, a_n) = 0$ *if at least one of the elements* a_i *equals* 0. *Then A has the k-local approximation property for an arbitrary k iff for all non-trivial congruences the congruence class containing* 0 *is dense in A.*

Proof. All we have to show is that the assumptions imply that the conditions of Theorems 2 and 3 are satisfied.

(1) and (2) of Theorem 2 are fulfilled by putting $\eta = \varphi$, $\mu = \psi$. Now let κ be an arbitrary congruence relation, $I = \{a \in A, a\, \kappa\, 0\}$ and $i_1,\dots, i_n \in I$. Since polynomial functions are congruence preserving, that is $a\, \kappa\, b \Rightarrow \alpha(a)\, \kappa\, \alpha(b)$ for all $\alpha \in P_n(A)$, n arbitrary (see, e.g., proof of Proposition 1.11.3 in [12]) we obtain

$$\psi(i_j, a_j)\, \kappa\, \psi(0, a_j) = a_j, \qquad j = 1,\dots, n, \text{ for all } a_1,\dots, a_n \in A.$$

Thus for an arbitrary n-ary operation ($n \geqslant 1$), $\omega \in \Omega$, we have

$$\omega(\psi(i_1, a_1),\dots, \psi(i_n, a_n))\, \kappa\, \omega(a_1,\dots, a_n).$$

Taking $a = 0$ in (2) we get $\varphi(b, b) = 0$ and using this we obtain

$$\begin{aligned} i := \varphi(\omega(\psi(i_1, a_1),\dots, \psi(i_n, a_n)), \omega(a_1,\dots, a_n))\,\kappa \\ \kappa\, \varphi(\omega(a_1,\dots, a_n), \omega(a_1,\dots, a_n)) = 0, \end{aligned}$$

therefore $i \in I$. Finally using (2) again we derive

$$\begin{aligned}
&\omega(\psi(i_1, a_1),\ldots, \psi(i_n, a_n)) \\
&\quad = \psi(\varphi(\omega(\psi(i_1, a_1),\ldots, \psi(i_n, a_n)), \omega(a_1,\ldots, a_n)), \omega(a_1,\ldots, a_n)) \\
&\quad = \psi(i, \omega(a_1,\ldots, a_n))
\end{aligned}$$

which shows that condition (1) of Theorem 2 is satisfied too.

Concerning the conditions of Theorem 3 we already have proved that (1) is satisfied. The others are fulfilled automatically.

Restricting ourselves to the finite case we get the following corollary which is similar to Theorem 2 in [6].

COROLLARY 2. *A finite universal algebra A is polynomially complete iff it has the following properties*:

(1) *A is simple.*

(2) *There are $0 \in A$, $\varphi, \psi \in P_2(A)$ and $\chi \in P_n(A)$, $n \geqq 2$, such that*

(α) *$\varphi(a, a) = 0$ for all $a \in A$.*

(β) *$\psi(a, 0) = \psi(0, a) = a$ for all $a \in A$.*

(γ) *$\psi(\varphi(a, b), b) = a$ for all $a, b \in A$.*

(δ) *χ is not constant and*

(ε) *$\chi(a_1,\ldots, a_n) = 0$ if at least one of the elements $a_i = 0$.*

Proof. If A is polynomially complete, then every $f \in F_k(A)$ is a polynomial function ($k \geqq 1$). Now it is easy to check that conditions (α) to (γ) are consistent, which means that there exist functions $\varphi, \psi \in F_2(A)$ satisfying these conditions. Thus (2) holds. Property (1) follows from Proposition 1.11.3 in [12]. Using Remark 3, the reverse statement is a special case of Theorem 3.

As further application we get:

THEOREM 5. *A topological T_2 group G has the local approximation property iff G is non-abelian and all normal subgroups $\neq E$ of G are dense in G.*

Proof. If G has the local approximation property, it also has the 1-local approximation by Proposition 1 and we can apply Folgerung 2 of [10]—the group S_2 with two elements cannot have the local approximation property because of Remark 3 and Theorem 1.12.53 of [12]. If, on the other hand, $G = \langle G, +, -, 0\rangle$ is non-abelian such that all normal subgroups $\neq E$ all normal subgroups $\neq E$ of G are dense in G, then we can choose $\varphi(a, b) = a - b$, $\psi(a, b) = a + b$, $\chi(a, b) = a + b - a - b$, and apply Theorem 4.

THEOREM 6. *A topological T_2 ring R has the local approximation property iff R is not a zero-ring and all ideals $\neq \{0\}$ are dense in R.*

Proof. One direction follows by using Folgerung 3 of [10]—the case where R is the zero-ring of order 2 can be excluded by applying the Theorem in [8] (of course every locally complete topological universal algebra has the local approximation property). The other direction can be derived by Theorem 4 putting $\varphi(a, b) = a - b$, $\psi(a, b) = a + b$ and $\chi(a, b) = ab$.

Non-trivial examples (that are those which are not locally complete) to Theorem 5 and 6 can be found in [10].

Next we want to apply Theorem 3 to the variety of loops $L = \langle L, +, /, \backslash, 0\rangle$, where $+$ denotes addition, $/$ left- and $\backslash$ right-subtraction, respectively, and 0 is the identity with respect to $+$ (see Bruck [3]). Topological loops have been studied in various papers, e.g., by Hofmann [4], and play an important role in topological investigations of geometry.

THEOREM 7. *A topological T_2 loop L has the local approximation property iff L is non-abelian or non-associative and if for all non-trivial congruences on L the congruence class containing* 0 *is dense in L.*

Proof. Since associative loops are groups, by Theorem 5 we need only consider the non-associative case (the polynomial functions considered either in the variety of associative loops or in the variety of groups coincide evidently). Here the assumptions of Theorem 4 are fulfilled, thus the if-part of Theorem 7 is true.

Conversely, we take $\varphi(a, b) = a/b$, $\psi(a, b) = a + b$, and $\chi(a, b, c) = (a + (b + c))/((a + b) + c)$. Then all conditions of Theorem 4 are fulfilled and application of this theorem proves the only if-part.

Finally, we want to deal with the variety of near-rings. We recall that a near-ring is a universal algebra $N = \langle N, +, -, 0, \cdot\rangle$ with two binary operations $+$, $\cdot$, one unary operation $-$, and one nullary operation 0 such that (i) $\langle N, +, -, 0\rangle$ is a (not necessarily abelian) group, (ii) $\langle N, \cdot\rangle$ is a semigroup and (iii) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in N$. An ideal I of N is a normal subgroup of $\langle N, +, -, 0\rangle$, which satisfies furthermore $N \cdot I \subseteq I$ and $(a + i) \cdot b - a \cdot b \in I$, $i \in I$, $a, b \in N$. For details we refer to [14].

THEOREM 8. *Let N be a topological T_2 near-ring. Then N has the local approximation property iff*

(1) *all ideals $\neq \{0\}$ are dense in N,*

(2) $\langle N, +, -, 0\rangle$ *is non-abelian, or*

(3) $\langle N, +, -, 0\rangle$ *is abelian and the multiplication $\cdot$ depends on both arguments.*

Proof. Suppose N to have the local approximation property. Corollary 1 yields condition (1). Now assume that $\langle N, +, -, 0\rangle$ is abelian and the multiplication does not depend on the second argument. Then $a \cdot b = a \cdot 0 = 0$; hence N is a zero-ring, which contradicts Theorem 6. If, on the other hand, multiplication does not depend on the first argument, then $a \cdot b = 0 \cdot b$, $a, b \in N$; thus the function $\nu(a) = 0 \cdot a$ is an endomorphism of the near-ring N. If $\ker \nu \neq \{0\}$, then Corollary 1 implies that $\ker \nu$ is dense in N. Since ν is continuous and N is T_2, we can apply the principle of extension of identities yielding $\ker \nu = N$; thus N is again a zero-ring, a contradiction.

There remains only the case $\ker \nu = \{0\}$, which means that ν is an automorphism of N. The equation $(0 \cdot c) \cdot (0 \cdot d) = 0 \cdot c \cdot d = 0 \cdot d$, $c, d \in N$, then implies that the multiplication is given by $a \cdot b = b$, $a, b \in N$. In that case, the polynomial functions of N to N evidently have the form $p(x) = a + nx$, where $a \in N$ and n is an arbitrary integer. Now we can carry over the proof of Folgerung 3 in [10] and see that N cannot have the 1-local and thus not the local approximation property, a contradiction.

Conversely, assume that (1) and (2) or (3) hold and set $\varphi(a, b) = a - b$, $\psi(a, b) = a + b$ and $\chi(a, b) = a + b - a - b$ in case (2), or $\chi(a, b) = a \cdot b - 0 \cdot b$ in case (3), respectively. Then all conditions of Theorem 4 are satisfied, which implies that N has the local approximation property as claimed.

EXAMPLE. (For the definitions of the concepts occurring in the following we refer to [1, Sect. 1].) Let N be an arbitrary near-ring with identity which is not a ring, let Γ be an N-group in the usual sense (see, for example [5, Sect. 10]) such that N is 2-primitive on Γ. Denote by $G = \mathrm{Aut}_\Gamma N$, the group of N-automorphisms of Γ, and by $M_G(\Gamma)$ the near-ring defined by $M_G(\Gamma) = \{f : \Gamma \to \Gamma,\ f \circ m = m \circ f \text{ for all } m \in G\}$. The sets $S(m, \gamma) = \{f \in M_G(\Gamma), f(\gamma) = m(\gamma)\}$, $\gamma \in \Gamma$, $m \in G$, form a subbasis of the so-called finite topology $\mathfrak{F}$ on $M_G(\Gamma)$ and $\langle M_G(\Gamma), \Omega, \mathfrak{F}\rangle$ becomes a topological near-ring with N dense in it. It is easy to show that $M_G(\Gamma)$ is T_2 by proving first that it is T_1 and then using the group structure of $M_G(\Gamma)$ this implies T_2. Now Proposition 3.3 in [1] yields that all invariant subnear-rings, thus in particular all ideals, of N ($M_G(\Gamma)$, respectively) are dense in $M_G(\Gamma)$. Therefore N ($M_G(\Gamma)$) has the local approximation property, if we choose $\langle N, +, -, 0\rangle$ non-abelian to satisfy condition 2 of Theorem 8. If we furthermore assume N ($M_G(\Gamma)$) to be not simple—which always is possible—, we get that N ($M_G(\Gamma)$) is not locally complete (see Introduction in [8]).

4. A Jacobson Density Theorem

Let K be a skewfield, V an arbitrary vectorspace over K, and denote by $L(V, V)$ the ring of all linear transformations of V into V. Endow V with the discrete topology, V^V with the product topology, and $L(V, V)$ with the induced topology. Then $L(V, V)$ is a topological T_2 ring and Jacobson's density theorem holds ([7, 11.4, Theorem 4]), which we will prove by means of the local approximation property:

Theorem 9 (A Jacobson density theorem). *Let R be a dense subring of $L(V, V)$. Then any ideal I of R is dense in $L(V, V)$.*

Proof. We will show directly that $L(V, V)$ has the 1-local approximation property which by Theorems 1 and 5 suffices to prove the theorem. Let $\varphi_1, ..., \varphi_n$ be fixed different elements of $L(V, V)$, and U be an arbitrary neighborhood of the identity function. Then by Folgerung 1 in [10] we have to find a polynomial function $p: L(V, V) \to L(V, V)$ such that $p(\varphi_1) \in U$ and $p(\varphi_i) = 0$ for $i = 2, ..., n$. Now $p(\varphi_1) \in U$ means: given arbitrary m different elements $x_1, ..., x_m \in V$, then $p(\varphi_1)(x_j) = x_j$ for $j = 1, ..., m$. Because of the linearity of $p(\varphi_1)(x_j) = x_j$ for $j = 1, ..., m$. Because of the linearity of $p(\varphi_1)$ we may assume that the x_j are linearly independent. We put $p_i = \varphi_1 - \varphi_i$, $i = 2, ..., n$. Then $p_i \in L(V, V)$ and $p_i \neq \mathrm{o}$, the zero-function. Let i, j be arbitrary but fixed indices, $j \in \{1, ..., m\}$, $i \in \{2, ..., n\}$.

(1) $x_j \in \ker p_i$. Let B be a basis of V containing $x_1, ..., x_m$. Because $p_i \neq \mathrm{o}$, there exists a $b \in B$ with $p_i(b) \neq 0$. Defining the linear transformation τ_{ij} by $\tau_{ij}(x_j) = b$, $\tau_{ij}(b) = x_j$ and $\tau_{ij}(a) = a$ for all $a \in B$, $a \neq x_j$, b we get $q_{ij} := p_i \circ \tau_{ij} \in L(V, V)$ and $q_{ij}(x_j) \neq 0$, which shows that we can restrict ourselves to the second alternative:

(2) $x_j \notin \ker p_i$ ($x_j \notin \ker q_{ij}$, respectively). Let C be a basis of $\ker p_i$ and extend this to a basis D of V containing x_j. Evidently the set $\{p_i(a),\ a \in D - C\}$ is linearly independent, so we can extend it to a basis F of V. Define $\rho_{ij} \in L(V, V)$, transforming the basis F into the basis D, by $\rho_{ij}(p_i(x_j)) = x_j$ and arbitrarily otherwise, and finally define $\sigma_{ij} \in L(V, V)$ by $\sigma_{ij}(x_j) = x_j$, $\sigma_{ij}(d) = 0$ for $d \in D$, $d \neq x_j$. Then consider the polynomial function p defined by

$$p(x) = \sum_{j=1}^{m} \prod_{i=2}^{n}{}^{*} \sigma_{ij} \circ \rho_{ij} \circ (x - \varphi_i) \circ \tau_{ij}^{*},$$

where $\circ$ means the composition of functions, $\prod^*$ denotes that the product is meant with respect to this composition, and τ_{ij}^* means that we only have to add this coefficient if $x_j \in \ker p_i$. Of course, p is a polynomial function on $L(V, V)$. Furthermore $p(\varphi_i) = 0$ for $i = 2, ..., n$ since $x - \varphi_i$ occurs in every summand. Finally, calculating $p(\varphi_1)(x_k)$, we first remark that only the summand with index k has to be considered since the others take the value 0 because of the definition of σ_{ij}. Therefore, $p(\varphi_1)(x_k) = \prod^*{}_{i=2}^{n} (\sigma_{ik} \circ \rho_{ik} \circ (\varphi_1 - \varphi_i) \circ \tau_{ik}^*)(x_k)$. Each of these factors sends x_k to x_k, so combining we really get $p(\varphi_1)(x_k) = x_k$ for all $k = 1,..., m$, what we have claimed.

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